Critical exponents of the degenerate Hubbard model

1993 J. Phys. A: Math. Gen. 26 1463

(http://iopscience.iop.org/0305-4470/26/7/009)

J. Phys. A: Math. Gen. 26 (1993) 1463–1480. Printed in the UK

Critical exponents of the degenerate Hubbard model

Holger Frahm† and Andreas Schadschneider‡§

† Institut für Theoretische Physik, Universität Hannover, D-3000 Hannover 1, Federal Republic of Germany

‡ Institut für Theoretische Physik, Universität zu Köln, D-5000 Köln 41, Federal Republic of Germany

Received 20 July 1992, in final form 30 November 1992

Abstract. We study the critical behaviour of the SU(N) generalization of the one-dimensional Hubbard model with arbitrary degeneracy N. Using the integrability of this model by Bethe ansatz we are able to compute the spectrum of the low-lying excitations in a large but finite box for arbitrary values of the electron density and of the Coulomb interaction. This information is used to determine the asymptotic behaviour of correlation functions at zero temperature in the presence of external fields lifting the degeneracy. The critical exponents depend on the system parameters through an $N \times N$ dressed charge matrix implying the relevance of the interaction of charge- and spin-density waves.

1. Introduction

The physics of highly correlated electron systems has long been the subject of extensive studies in condensed matter physics. Recently, the non-Fermi liquid character of low-dimensional systems has attracted renewed interest in one-dimensional realizations of these systems where large quantum fluctuations lead to Luttinger liquid behaviour [1]: the correlation functions decay as power laws at zero temperature, the exponents depending on the system parameters such as electron density, magnetization (or applied magnetic field) and strength of the interaction.

In this context exactly soluble models can provide a variety of new insights, in particular when used together with the general results on quantum critical behaviour in one spatial dimension as provided by the theory of conformal invariance [2–4]. Here the universality class of the quantum system is completely determined by a single dimensionless number—the so-called central charge c of the underlying Virasoro algebra. This number c as well as the dimensions of the operators present in the theory can be extracted from analytical results for the spectrum of low-lying states in finite geometries. In this language, Luttinger liquids correspond to a central charge $c = 1$, the dependence of the anomalous dimensions on the system parameters is through a single dimensionless number—the coupling constant of the corresponding Gaussian model.

† e-mail: frahm@kastor.itp.uni-hannover.de

‡ e-mail: as@thp.uni-koeln.de

§ Work performed within the research programme of the Sonderforschungsbereich 341 (Köln–Aachen–Jülich).

0305-4470/93/071463+18$07.50

The situation described above is the generic behaviour of one-dimensional (1D) quantum systems with a *single* critical degree of freedom, as realized in spin-$\frac{1}{2}$ chains or systems of spinless fermions. The situation becomes more complicated for systems where two or more massless excitations are possible: the lack of Lorentz invariance (the corresponding Fermi velocities differ in general) prevents the direct application of the predictions of conformal field theory and the interacting nature of the system complicates the factorization of the problem into *independent* ones for each critical quasi-particle mode. On the other hand the exact results on the finite-size scaling of the low-lying energies available for Bethe ansatz soluble models suggests a resolution of this problem: the spectrum is that of a multi-component Gaussian model. In analogy to the analysis of conformal invariant theories the universality class of a system with N massless collective excitation modes is determined by N dimensionless numbers c_r—reducing to the central charge in the scalar case. The anomalous dimensions are functions of the system parameters not through a single coupling constant but through an $N \times N$ matrix of dimensionless numbers—the so-called dressed charge matrix. This behaviour has been found in a large number of 1D quantum systems, including certain integrable spin-chains with $S > \frac{1}{2}$ [5,6], and the Hubbard [7–9] and t–J models of correlated electrons, the latter both at the integrable supersymmetric point $J/t = 2$ [10] and away from integrability [11].

Most of the integrable models in this list are solved by a hierarchy of Bethe ansätze [12]: the first one introduces a set of wavenumbers describing the phase of the wavefunction and determining the spectrum, the others are necessary for the wavefunctions to show the symmetry corresponding to a particular representation of the permutation group. This allows for the solution of certain systems with various choices of internal degrees of freedom, e.g. an SU(2)-spin in the Hubbard model.

On the level of the Bethe ansatz equations—which have to be solved for the calculation of the spectrum of the model—this opens the possibility for studying a generalization of the Hubbard model which, instead of spin-$\frac{1}{2}$ electrons, describes interacting fermions carrying an SU(N)-spin index. A Hamiltonian for this *degenerate* Hubbard model has been proposed previously by various authors [13–17]:

$$\mathcal{H}_N = -\sum_{j=1}^{L}\sum_{s=1}^{N}\mathcal{P}(c_{j+1,s}^{\dagger}c_{js} + c_{js}^{\dagger}c_{j+1,s})\mathcal{P} + 4u\sum_{j=1}^{L}\sum_{s,s'(s\neq s')} n_{js}n_{js'} + \mu\sum_{j=1}^{L}\sum_{s=1}^{N} n_{js} + \sum_{j=1}^{L}\sum_{s=1}^{N} h_s n_{js}. \tag{1.1}$$

Here, the Fermi operator $c_{js}^{\dagger}$ (c_{js}) creates (annihilates) an electron at site j with spin index $s \in \{1,\ldots,N\}$ and $n_{js} = c_{js}^{\dagger}c_{js}$ is the corresponding number operator. The real parameters h_s may be considered as generalized magnetic fields. $\mathcal{P}$ projects onto the subspace of states having at most two electrons at each site.

This projection is crucial for the applicability of the Bethe ansatz to this model as it prevents scattering processes involving three or more electrons on one site [18]. For spin-$\frac{1}{2}$ electrons these configurations are excluded automatically by the Pauli principle—the operator $\mathcal{P}$ does not impose any additional restrictions. However, as was realized later, the projectors do not solve the problem of many-electron scattering processes: the wavefunctions resulting from the analysis do *not* satisfy this constraint. In fact, in the limit $u \to 0$ the Bethe ansatz equations for the

SU(N) model describe a system of *free* fermions with N components which is in contradiction to the correlations introduced in equation (1.1) through the projection operators $\mathcal{P}$. Only recently the question of which model is solved by the Bethe ansatz was addressed again by Schlottmann [19] who has shown that in the continuum limit the scattering phase shifts as found from the Bethe ansatz equations are those of particles interacting via a potential of the form $1/\sinh^2 r$, where r is the particle distance in suitably chosen units. This indicates the need for a long-range interaction dynamically excluding three-electron configurations for finite u. However, the correct form of the interaction for the lattice model has not yet been found.

There exist various limits in which the problem stated above can be resolved [19]: in the continuum limit and for small densities or $u \gg 1$ in the lattice model the contribution of unwanted configurations becomes negligible, so the Hamiltonian of the form (1.1) describes the system reasonably well.

However, despite the problem outlined above, it appears to be worthwhile to study the the SU(N) lattice model defined by the Bethe ansatz equations given below—even beyond these limiting cases:

(i) the SU(N) model allows phenomena not found in the $N = 2$ model to be studied (from previous studies the system is known to undergo a Mott transition at a *finite* value of the Coulomb interaction [13,20,21]);

(ii) an extensive study of the properties of this model provides the necessary information as one tries to construct the corresponding Hamiltonian;

(iii) exact results, e.g. for the behaviour of correlation functions in models of this type, can be used to check predictions derived from asymptotic $1/N$ expansions [22].

In the present paper we investigate the asymptotic behaviour of correlation functions in the degenerate Hubbard model.

Our paper is organized as follows. In the following section we shall introduce the Bethe ansatz equations describing electrons carrying an SU(N)-spin index and give a qualitative discussion of the resulting excitation spectrum. In section 3 the results for the finite-size corrections to the energies of low-lying states as well as their relation to the critical exponents are given in terms of the $N \times N$ dressed charge matrix. In section 4 the integral equations for this matrix are solved in the zero-field case and the operator dimensions are computed as a function of the electron density and the strength of the interaction. They are shown to reflect the full SU(N)-spin symmetry present in this case. Furthermore, we discuss the variation of the operator dimensions as the Mott transition mentioned above is approached. These results are applied to the computation of the critical exponents for some correlation functions of interest. Finally, in section 5 we consider states where the SU(N) symmetry of the ground state is broken by magnetic fields coupling to the various flavours of the internal degree of freedom as in (1.1). In the limiting case of strong coupling we discuss the dependence of the dressed charge matrix and the critical exponents on these external fields.

2. The Bethe ansatz solution of the SU(N) Hubbard model

The Bethe ansatz wavefunction which solves the Schrödinger equation for the SU(N) generalization of the Hubbard model on a chain of length L for a total number N_c of electrons is characterized by the momenta k_j ($j = 1, \ldots, N_c$) and $N - 1$ sets of rapidities $\lambda_\alpha^{(s)}$ ($s = 1, \ldots, N-1$; $\alpha = 1, \ldots, M_s$). Imposing periodic boundary

conditions on the wavefunction leads to Bethe ansatz equations [13, 14, 17]

$$Lk_j = 2\pi I_j - \sum_{\beta=1}^{M_1} 2\tan^{-1}\left(\frac{\sin k_j - \lambda_\beta^{(1)}}{u}\right)$$

$$\sum_{\beta=1}^{M_{s-1}} 2\tan^{-1}\left(\frac{\lambda_\alpha^{(s)} - \lambda_\beta^{(s-1)}}{u}\right) + \sum_{\beta=1}^{M_{s+1}} 2\tan^{-1}\left(\frac{\lambda_\alpha^{(s)} - \lambda_\beta^{(s+1)}}{u}\right)$$

$$= 2\pi J_\alpha^{(s)} + \sum_{\beta=1}^{M_s} 2\tan^{-1}\left(\frac{\lambda_\alpha^{(s)} - \lambda_\beta^{(s)}}{2u}\right). \tag{2.1}$$

Here we have set $M_0 = N_c$, $M_N = 0$ and $\lambda_j^{(0)} = \sin k_j$.

The quantum numbers I_j and $J_\alpha^{(s)}$ are integer or half-integer depending on the parity of the numbers N_c, $M_\alpha^{(s)}$:

$$I_j = \frac{M_1}{2} \bmod 1 \qquad J_\alpha^{(s)} = \frac{M_s - M_{s-1} - M_{s+1} + 1}{2} \bmod 1. \tag{2.2}$$

Energy and momentum of the model in a state corresponding to a solution of (2.1) are completely determined by the momenta k_j:

$$E = -2\sum_{j=1}^{N_c} \cos k_j + \mu N_c - \sum_{s=1}^{N} h_s N_s$$

$$P = \sum_{j=1}^{N_c} k_j = \frac{2\pi}{L}\left(\sum_{j=1}^{N_c} I_j + \sum_{s=1}^{N-1}\sum_{\alpha=1}^{M_s} J_\alpha^{(s)}\right) \tag{2.3}$$

where N_s denotes the total number of electrons with orbital index s.

In the following we consider the Bethe ansatz equations (2.1) of the SU(N) Hubbard model coupled to generalized magnetic fields h_s as in (1.1). In the thermodynamic limit ($L \to \infty$, with N_c/L, M_s/L kept constant) the equations (2.1) corresponding to the ground state can be transformed into a set of coupled integral equations for the densities $\rho_c(k)$ and $\rho_s(\lambda)$ of the parameters k_j and $\lambda_\alpha^{(s)}$, respectively,

$$\rho_c(k) = \frac{1}{2\pi} + \frac{\cos k}{2\pi}\int_{-\Lambda_1}^{\Lambda_1} \mathrm{d}\lambda\, K_1(\sin k - \lambda)\rho_1(\lambda)$$

$$\rho_1(\lambda) = \frac{1}{2\pi}\int_{-k_0}^{k_0} \mathrm{d}k\, K_1(\lambda - \sin k)\rho_c(k) - \frac{1}{2\pi}\int_{-\Lambda_1}^{\Lambda_1} \mathrm{d}\mu\, K_2(\lambda - \mu)\rho_1(\mu)$$

$$+ \frac{1}{2\pi}\int_{-\Lambda_2}^{\Lambda_2} \mathrm{d}\mu\, K_1(\lambda - \mu)\rho_2(\mu) \tag{2.4}$$

$$\rho_s(\lambda) = \frac{1}{2\pi}\int_{-\Lambda_{s-1}}^{\Lambda_{s-1}} \mathrm{d}\mu\, K_1(\lambda - \mu)\rho_{s-1}(\mu) - \frac{1}{2\pi}\int_{-\Lambda_s}^{\Lambda_s} \mathrm{d}\mu\, K_2(\lambda - \mu)\rho_s(\mu)$$

$$+ \int_{-\Lambda_{s+1}}^{\Lambda_{s+1}} \mathrm{d}\mu\, K_1(\lambda - \mu)\rho_{s+1}(\mu) \qquad (s = 2, \ldots, N-1)$$

with $\Lambda_N = 0$. The kernels $K_{1,2}(x)$ of these equations (2.4) are given by

$$K_1(x) = 2u/x^2 + u^2 \qquad K_2(x) = 4u/x^2 + (2u)^2. \tag{2.5}$$

The values of the parameters k_0 and $\Lambda_1, \ldots, \Lambda_{N-1}$ are determined through the normalizations

$$n_c = \int_{-k_0}^{k_0} \mathrm{d}k\, \rho_c(k)$$

$$n_s = \int_{-\Lambda_{s-1}}^{\Lambda_{s-1}} \mathrm{d}\lambda\, \rho_{s-1}(\lambda) - \int_{-\Lambda_s}^{\Lambda_s} \mathrm{d}\lambda\, \rho_s(\lambda) \qquad (s = 1, \ldots, N-1) \tag{2.6}$$

where $n_c = N_c/L$ is the total density of electrons and $n_s = N_s/L = (M_{s-1} - M_s)/L$ is the density of electrons with index s. Furthermore we have set $\rho_0 \equiv \rho_c$ and $\Lambda_0 = k_0$†.

The ground-state energy per lattice site is

$$\epsilon_\infty = \int_{-k_0}^{k_0} \mathrm{d}k\, (\mu + h_1 - 2\cos k)\rho_c(k) + \sum_{s=1}^{N-1} (h_{s+1} - h_s) \int_{-\Lambda_s}^{\Lambda_s} \mathrm{d}\lambda\, \rho_s(\lambda) \tag{2.7}$$

which may alternatively be expressed in terms of the dressed energy

$$\epsilon_\infty = \frac{1}{2\pi} \int_{-k_0}^{k_0} \mathrm{d}k\, \varepsilon_c(k). \tag{2.8}$$

Here $\varepsilon_c(k)$ is the solution of the system of coupled integral equations

$$\varepsilon_c(k) = \varepsilon_c^{(0)}(k) + \frac{1}{2\pi} \int_{-\Lambda_1}^{\Lambda_1} \mathrm{d}\lambda\, K_1(\sin k - \lambda)\varepsilon_1(\lambda)$$

$$\varepsilon_1(\lambda) = \varepsilon_1^{(0)}(\lambda) + \frac{1}{2\pi} \int_{-k_0}^{k_0} \mathrm{d}k \cos k K_1(\lambda - \sin k)\varepsilon_c(k) - \frac{1}{2\pi} \int_{-\Lambda_1}^{\Lambda_1} \mathrm{d}\mu\, K_2(\lambda - \mu)\varepsilon_1(\mu)$$
$$+ \frac{1}{2\pi} \int_{-\Lambda_2}^{\Lambda_2} \mathrm{d}\mu\, K_1(\lambda - \mu)\varepsilon_2(\mu) \tag{2.9}$$

$$\varepsilon_s(\lambda) = \varepsilon_s^{(0)}(\lambda) + \frac{1}{2\pi} \int_{-\Lambda_{s-1}}^{\Lambda_{s-1}} \mathrm{d}\mu\, K_1(\lambda - \mu)\varepsilon_{s-1}(\mu) - \frac{1}{2\pi} \int_{-\Lambda_s}^{\Lambda_s} \mathrm{d}\mu\, K_2(\lambda - \mu)\varepsilon_s(\mu)$$
$$+ \frac{1}{2\pi} \int_{-\Lambda_{s+1}}^{\Lambda_{s+1}} \mathrm{d}\mu\, K_1(\lambda - \mu)\varepsilon_{s+1}(\mu) \qquad (s = 2, \ldots, N-1).$$

The bare energies are from (2.3)

$$\varepsilon_c^{(0)}(k) = \mu + h_1 - 2\cos k \qquad \varepsilon_s^{(0)}(\lambda) = h_{s+1} - h_s. \tag{2.10}$$

† In the following we shall adopt the convention that an index '0' stands for 'c'.

The dressed energies (2.9) obey the conditions

$$\varepsilon_c(k_0) = 0 \qquad \varepsilon_s(\Lambda_s) = 0. \tag{2.11}$$

The ground state at half-filling ($n_c = 1$) and with vanishing fields h_s shows for $N > 2$ an interesting behaviour which has not been found in the case of the standard Hubbard model ($N = 2$). This has been noticed independently by Schlottmann [20] and one of the authors in [21]. For $u > u_c$ one finds $k_0 = \pi$ and for $u < u_c$ one has $k_0 < \pi$. Here the critical value u_c is determined through the implicit equation

$$\int_{-\pi}^{\pi} \mathrm{d}k\, G_N(\sin k; u_c) = 2\pi \tag{2.12}$$

where $G_N(x; u)$ in terms of the digamma function $\psi(x)$ is given by

$$G_N(x; u) = \frac{1}{Nu} \operatorname{Re}\left[\psi\left(1 + \mathrm{i}\frac{x}{2Nu}\right) - \psi\left(\frac{1}{N} + \mathrm{i}\frac{x}{2Nu}\right)\right]. \tag{2.13}$$

For $N = 2$ we have $u_c = 0$ (see figure 1) as already shown by Lieb and Wu [23].

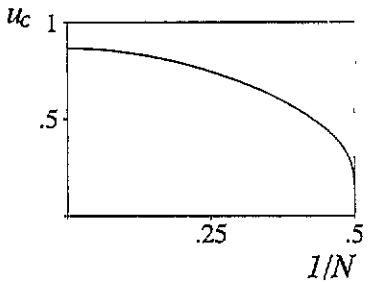

Figure 1. Dependence of the critical value u_c on the degeneracy N as obtained from (2.12). Note that for $u > u_c$ the system is in an insulating phase whereas for $u < u_c$ it shows metallic behaviour (for $n_c = 1$).

The excitation spectrum in zero fields has also been studied in [21] using an extension of the method developed in [24,25] and in [20]. One finds $N - 1$ gapless spin excitations with soft modes with wavenumbers $2sP_F$ ($s = 1, \ldots, N-1$, $P_F = (\pi/N)n_c$) and so-called particle–hole excitations†. These also are gapless and do exist only for $k_0 < \pi$, i.e. for $n_c < 1$ or $n_c = 1$ and $u < u_c$. In [21] also, excitations corresponding to doubly occupied sites have been studied. These are described by complex momenta $k^{\pm}$ satisfying $\sin k^{\pm} = \lambda \pm \mathrm{i}u$ and have a finite gap (at least for $n_c = 1$ and $u > u_c$).

The special structure of the ground state for $n_c = 1$ leads to interesting properties of the model. For $u > u_c$, the only possible charge-carrying excitations are those involving complex momenta. As these excitations have a gap the system is in an insulating state. For $u < u_c$, particle–hole excitations become possible. These excitations may carry a current and so the system is in a metallic phase. This shows the existence of a Mott transition at the critical value u_c of the Coulomb repulsion u. The transition is also reflected in the behaviour of other physical quantities, e.g. the charge susceptibility χ_c and the Fermi velocity v_c [20,21]. As the gap to the

† For a discussion of the zero-field excitation spectrum for $N = 2$, see e.g. [24] and references therein.

excitations with double occupations does not vanish in the limit $u \searrow u_c$ the value of the gap to the charge-carrying shows a discontinuity at $u = u_c$. We may thus say that the transition is of 'first order'.

In the limit $u \to \infty$ at $n_c = 1$ the SU(N) Hubbard model becomes equivalent to the SU(N) Heisenberg chain [26]. This equivalence generalizes the well known relation between the regular ($N = 2$) Hubbard model and the Heisenberg antiferromagnet.

3. Finite-size corrections and conformal properties

As shown in the preceding section the degenerate Hubbard model supports gapless excitations in general. Thus we may apply the concepts of conformal field theory to determine the asymptotic behaviour of the correlation functions, e.g. the critical exponents.

First we calculate exactly the finite-size corrections to the ground state energy and the energies of the excited states. This can be done by a straightforward extension of the calculation for the case $N = 2$ [7]. The results can be expressed in terms of the $N \times N$ dressed charge matrix

$$Z = \begin{pmatrix} \xi_{cc}(k_0) & \xi_{c1}(\Lambda_1) & \cdots & \xi_{c,N-1}(\Lambda_{N-1}) \\ \xi_{1c}(k_0) & \xi_{11}(\Lambda_1) & \cdots & \xi_{1,N-1}(\Lambda_{N-1}) \\ \vdots & \vdots & & \vdots \\ \xi_{N-1,c}(k_0) & \xi_{N-1,1}(\Lambda_1) & \cdots & \xi_{N-1,N-1}(\Lambda_{N-1}) \end{pmatrix}. \tag{3.1}$$

The elements of Z can be obtained from the dressed charge functions $\xi_{rs}(\lambda)$ which obey a system of coupled integral equations similar to (2.9). For $r = c, 1, \ldots, N-1$ we have

$$\xi_{rc}(k) = \delta_{rc} + \frac{1}{2\pi}\int_{-\Lambda_1}^{\Lambda_1} \mathrm{d}\lambda \cos k K_1(\sin k - \lambda)\xi_{r1}(\lambda)$$

$$\xi_{r1}(\lambda) = \delta_{r1} + \frac{1}{2\pi}\int_{-k_0}^{k_0} \mathrm{d}k \cos k\, K_1(\lambda - \sin k)\xi_{rc}(k) - \frac{1}{2\pi}\int_{-\Lambda_1}^{\Lambda_1} \mathrm{d}\mu\, K_2(\lambda - \mu)\xi_{r1}(\mu)$$
$$+ \frac{1}{2\pi}\int_{-\Lambda_2}^{\Lambda_2} \mathrm{d}\mu\, K_1(\lambda - \mu)\xi_{r2}(\mu) \tag{3.2}$$

$$\xi_{rs}(\lambda) = \delta_{rs} + \frac{1}{2\pi}\int_{-\Lambda_{s-1}}^{\Lambda_{s-1}} \mathrm{d}\mu\, K_1(\lambda - \mu)\xi_{r,s-1}(\mu) - \frac{1}{2\pi}\int_{-\Lambda_s}^{\Lambda_s} \mathrm{d}\mu\, K_2(\lambda - \mu)\xi_{rs}(\mu)$$
$$+ \frac{1}{2\pi}\int_{-\Lambda_{s+1}}^{\Lambda_{s+1}} \mathrm{d}\mu\, K_1(\lambda - \mu)\xi_{r,s+1}(\mu) \qquad (s = 2, \ldots, N-1).$$

The finite-size scaling behaviour of the ground-state energy is found to be

$$E_0 - L\epsilon_\infty = -\frac{\pi}{6L}\sum_{s=0}^{N-1} v_s \tag{3.3}$$

with the Fermi velocities of charge and spin excitations

$$v_0 \equiv v_c = \frac{1}{2\pi\rho_c(k_0)}\varepsilon_c'(k_0) \qquad v_s = \frac{1}{2\pi\rho_s(\Lambda_s)}\varepsilon_s'(\Lambda_s) \qquad (s = 1, \ldots, N-1). \tag{3.4}$$

Energies and momenta of the excitations scale as

$$\begin{aligned} E(\Delta M, \boldsymbol{D}) - E_0 &= \frac{2\pi}{L}\left[\frac{1}{4}\Delta M^{\mathrm{T}}(Z^{-1})^{\mathrm{T}} V Z^{-1}\Delta M \right. \\ &\quad \left. + \boldsymbol{D}^{\mathrm{T}} Z V Z^{\mathrm{T}} \boldsymbol{D} + \sum_{s=0}^{N-1} v_s(N_s^+ + N_s^-)\right] \\ P(\Delta M, \boldsymbol{D}) - P_0 &= \frac{2\pi}{L}\left[\Delta M^{\mathrm{T}} \cdot \boldsymbol{D} + \sum_{s=0}^{N-1}(N_s^+ - N_s^-)\right] + 2\sum_{s=0}^{N-1}\sum_{r=0}^{s} D_r P_{\mathrm{F},s+1} \end{aligned} \tag{3.5}$$

$V = \mathrm{diag}(v_c, v_1, \ldots, v_{N-1})$.

Here $N_c^{\pm}$, $N_s^{\pm}$ are positive integers and ΔM and $\boldsymbol{D}$ are vectors characterizing the excited state under consideration. $P_{\mathrm{F},s}$ are the Fermi momenta for electrons with spin index s. For the ground state in the thermodynamic limit we have $\Delta M_s = 0$, $D_s = 0$ $(s = c, 1, \ldots, N-1)$. For an excited state ΔM has integer components denoting the change of the total number of electrons and the number of electrons with index s with respect to the ground state. D_s are integer or half-odd integer depending on the parities of the ΔM_s. Due to (2.2) we have

$$\begin{aligned} D_{c'} &= \frac{\Delta N_c + \Delta M_1}{2} \bmod 1 \\ D_s &= \frac{\Delta M_{s-1} + \Delta M_{s+1}}{2} \bmod 1 \qquad (s = 1, \ldots, N-1) \end{aligned} \tag{3.6}$$

with $\Delta M_0 = \Delta N_c$ and $\Delta M_N = 0$.

In general, all velocities v_s are different. In this case the results (3.3) and (3.5) may be interpreted in terms of a semidirect product of N independent Virasoro algebras†. All these Virasoro algebras have central charge $c_s = 1$. For vanishing fields h_s all magnon velocities $v_1, \ldots, v_{N-1}$ are equal [20,21] and we have a semidirect product of a $c = 1$ Gaussian theory—reflecting the U(1) symmetry of the charge sector—and a $c = N - 1$ Wess–Zumino–Witten theory—reflecting the SU(N) symmetry of the spin sector.

Comparing (3.5) with the predictions of the conformal field theory [3,4]

$$\begin{aligned} E(\Delta M, \boldsymbol{D}) - E_0 &= \frac{2\pi}{L}\sum_{s=0}^{N-1} v_s(\Delta_s^+ + \Delta_s^-) \\ P(\Delta M, \boldsymbol{D}) - P_0 &= \frac{2\pi}{L}\sum_{s=0}^{N-1}(\Delta_s^+ - \Delta_s^-) + 2\sum_{s=0}^{N-1}\sum_{r=0}^{s} D_r P_{\mathrm{F},s+1} \end{aligned} \tag{3.7}$$

† See [8] and references therein for a more detailed discussion of this point.

one obtains expressions for the conformal dimensions $\Delta_s^{\pm}$ of the primary fields in terms of the dressed charge matrix. Requiring that all dimensions are positive we find

$$2\Delta_s^{\pm} = ((Z^T \boldsymbol{D})_s \pm \tfrac{1}{2}(Z^{-1}\Delta M)_s)^2 + 2N_s^{\pm} \tag{3.8}$$

which, in general, depend on the system parameters. In the following section we shall show that for vanishing fields the $\Delta_s^{\pm}$ ($s = 1, \ldots, N-1$) are functions of the components of ΔM and $\boldsymbol{D}$ only, whereas $\Delta_c^{\pm}$ depends on the strength u of the Coulomb repulsion and the density n_e of electrons.

We now make further use of the results of conformal field theory to write down the correlation functions for primary fields as

$$\langle \phi_{\Delta^{\pm}}(x,t)\phi_{\Delta^{\pm}}(0,0)\rangle = \prod_{s=0}^{N-1} \frac{\exp(-2\mathrm{i}\sum_{r=0}^{s} D_r P_{\mathrm{F},s+1}x)}{(x - \mathrm{i}v_s t)^{2\Delta_s^{+}}(x + \mathrm{i}v_s t)^{2\Delta_s^{-}}}. \tag{3.9}$$

The correlation functions of the physical fields consist of a sum of terms (3.9). In the following we shall study correlators of the form $\langle \mathcal{O}_j(t)\mathcal{O}_0^{\dagger}(0)\rangle$ where $\mathcal{O}$ is given in terms of c and $c^{\dagger}$. To find the asymptotic behaviour of the correlator one has to expand $\mathcal{O}$ in terms of the conformal fields. This is not possible in general, but the explicit form of $\mathcal{O}$ allows for an identification of the quantum numbers $M_c, M_1, \ldots, M_{N-1}$ of the intermediate states. Therefore the leading term in the asymptotic expansion of $\langle \mathcal{O}_j(t)\mathcal{O}_0^{\dagger}(0)\rangle$ can be obtained from (3.8) through minimizing with respect to the D_s satisfying (3.6).

4. Critical exponents for vanishing fields

In the absence of magnetic fields it is easily seen that $\Lambda_r = \infty$ for all r. This allows the elimination of λ-dependent quantities from the Bethe ansatz integral equations by Fourier transformation. From equation (3.2) we obtain for the k-dependent entries of the dressed charge matrix

$$\xi_{rc}(z) = \frac{N-r}{N} + \frac{1}{2\pi}\int_{-z_0}^{z_0} \mathrm{d}y\, \xi_{rc}(y) G_N(z-y;1) \tag{4.1}$$

where we have introduced a new variable $z = \sin k/u$.

The solution for $z_c = \xi_{cc}(z_0)$ is obtained by iteration for small values of z_0 where we obtain

$$z_c \simeq 1 + [G_N(0;1)]/\pi z_0 = 1 - [\gamma + \psi(1/N)]/N\pi z_0 \qquad \text{for } z_0 \ll 1. \tag{4.2}$$

For $z_0 \gg N$ a perturbative scheme [27] based on the Wiener–Hopf method can be applied, giving

$$z_c \simeq \sqrt{N}(1 - (N-1)/2\pi z_0) \qquad \text{for } z_0 \gg N. \tag{4.3}$$

For intermediate values of z_0 the integral equation (4.1) is easily solved numerically. The dependence of z_c on the density and the strength of the Coulomb interaction is

shown in figure 2 for some values of N. In addition to the interpretation of the z_c as measure of the reordering of the Fermi sea due to the interaction when an electron is added there exists a direct relation to physical observables: it can be expressed as

$$z_c^2 = \pi v_c n_c^2 \kappa \tag{4.4}$$

in terms of the compressibility $\kappa = -(1/L)\partial L/\partial p$ of the electron gas (p being the pressure).

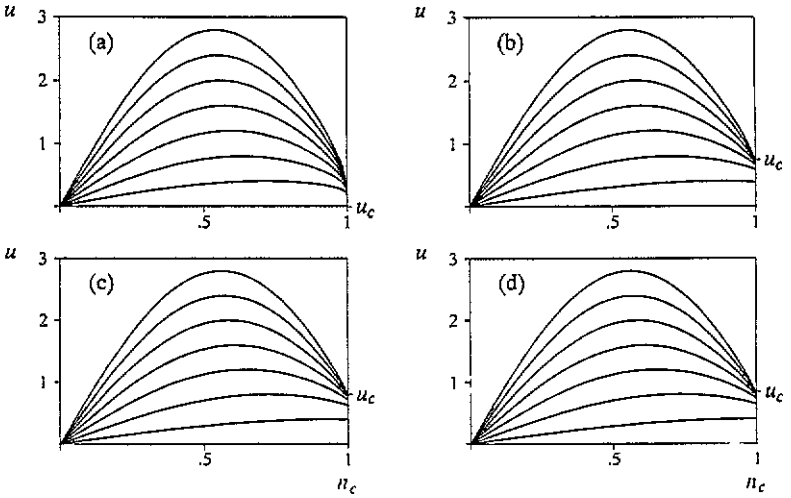

Figure 2. (a) Lines of constant z_c for vanishing magnetic fields in the n_c–u plane for $N = 2$. The drawn lines correspond to $z_c = 1.0904, 1.1679, 1.2237, 1.2622, 1.2892, 1.3088, 1.3234$ (corresponding to $z_0 = 0.4, 0.8, \ldots, 2.8$). Note that $z_c \to 1$ for $u \to 0$ and $z_c \to \sqrt{N}$ for $u \to \infty$ with n_c arbitrary, $n_c \to 0$ with u arbitrary, and $n_c \to 1$ with $u > u_c$. (b) As in (a) but for $N = 4$, $z_c = 1.1241, 1.2445, 1.3473, 1.4318, 1.5013, 1.5586, 1.6062$. (c) As in (a) but for $N = 6$, $z_c = 1.1316, 1.2630, 1.3799, 1.4812, 1.5689, 1.6453, 1.7122$. (d) As in (a) but for $N = 8$, $z_c = 1.1344, 1.2701, 1.3928, 1.5012, 1.5973, 1.6830, 1.7601$.

As in the regular Hubbard model ($N = 2$) [7] one can employ the Wiener–Hopf method to compute the remaining elements of the dressed charge matrix (3.1) yielding

$$Z = \begin{pmatrix} z_c & 0 & 0 & \cdots & 0 \\ ((N-1)/N)z_c & & & & \\ ((N-2)/N)z_c & & & & \\ \vdots & & \mathcal{Z}_N & & \\ 1/Nz_c & & & & \end{pmatrix}. \tag{4.5}$$

Note that the symmetric $(N-1) \times (N-1)$ block $\mathcal{Z}_N$ of this matrix is completely determined by the SU(N)-spin symmetry of the system: this symmetry is manifest in the kernel of the integral equations and allows for the reduction of the corresponding *matrix* Wiener–Hopf problem to $N-1$ *scalar* ones [28,29]. The latter are soluble by quadratures and one obtains closed expressions for the matrix elements of $\mathcal{Z}_N^{-1}$:

$$(\mathcal{Z}_N^{-1})_{rs} = -\frac{2}{N}\sin\frac{\pi}{2N}\frac{\sqrt{1-z_r^2}\sqrt{1-z_s^2}}{z_r^2+z_s^2-2\cos(\pi/2N)z_r z_s-\sin^2(\pi/2N)}$$
$$z_n = \cos\frac{\pi n}{N}. \tag{4.6}$$

The square of $\mathcal{Z}_N^{-1}$ is the Cartan matrix for the Lie algebra SU(N):

$$\left(\mathcal{Z}_N^{-2}\right)_{rs} = (\mathcal{C}_N)_{rs} = 2\delta_{r,s} - \delta_{r,s+1} - \delta_{r+1,s}.$$

Using the properties of the dressed charge matrix we find for the critical exponents

$$\begin{aligned} 2\Delta_c^{\pm} &= \left(z_c \sum_{s=0}^{N-1} \frac{N-s}{N} D_s \pm \frac{1}{2z_c} \Delta N_c \right)^2 + 2N_c^{\pm} \\ 2\Delta_r^{\pm} &= \left[\sum_{s=1}^{N-1} (\mathcal{Z}_N)_{sr} D_s \pm \frac{1}{2} \sum_{s=1}^{N-1} (\mathcal{Z}_N^{-1})_{rs} \left(\Delta M_s - \frac{N-s}{N} \Delta N_c \right) \right]^2 + 2N_r^{\pm}. \end{aligned} \tag{4.7}$$

For vanishing magnetic field the magnon velocities v_r ($r = 1, \ldots, N-1$) are identical, hence only the quantities

$$2\Delta_\sigma^{\pm} = 2 \sum_{r=1}^{N-1} \Delta_r^{\pm} = \tfrac{1}{4} \Delta \boldsymbol{N}_\sigma^{\mathrm{T}} \mathcal{C}_N \Delta \boldsymbol{N}_\sigma + \boldsymbol{D}_\sigma^{\mathrm{T}} \mathcal{C}_N^{-1} \boldsymbol{D}_\sigma \pm \Delta \boldsymbol{N}_\sigma^{\mathrm{T}} \cdot \boldsymbol{D}_\sigma + 2 \sum_{r=1}^{N-1} N_r^{\pm}$$

where

$$(\Delta \boldsymbol{N}_\sigma)_r = \Delta M_r - ((N-r)/N) \Delta N_c \qquad (\boldsymbol{D}_\sigma)_r = D_r \qquad r = 1, \ldots, N-1 \tag{4.8}$$

appear in the exponents describing the asymptotic behaviour of the correlation functions. Note that they are independent of the quantity z_c incorporating the dependence of the anomalous dimensions on electron density and strength of the interaction. They are completely determined by the SU(N) symmetry of the zero field ground state and, in fact, of the same form as the exponents characterizing the SU(N)-symmetric critical vertex models and spin chains [28,29] (the difference being the possibility of fractional values for the elements of $\Delta \boldsymbol{N}_\sigma$).

Now we are able to study the asymptotic behaviour of correlation functions of interest. For the field–field correlation function we have $\Delta N_c = 1$ and $\Delta M = (1, \ldots, 1_k, 0, \ldots)$. The corresponding values of $\boldsymbol{D}$ can be read from equation (3.6), the contribution at wavenumber $k = P_{\mathrm{F}} = (\pi/N) n_c$ arises from the choice $D_r = \frac{1}{2}(\delta_{rs} - \delta_{r,s+1})$ for any s giving

$$2\Delta_c^{\pm} = \left(\frac{z_c}{2N} \pm \frac{1}{2z_c} \right)^2 \qquad 2\Delta_\sigma^{+} = \left(1 - \frac{1}{N} \right) \qquad 2\Delta_\sigma^{-} = 0. \tag{4.9}$$

Hence, the singularity of the momentum distribution function at the Fermi point

$$\langle c_{k,s} c_{k,s}^{\dagger} \rangle \sim |k - P_{\mathrm{F}}|^{\alpha} \tag{4.10}$$

is characterized by the exponent

$$\alpha = \theta/[(2N)^2] + 1/\theta - 1/N \tag{4.11}$$

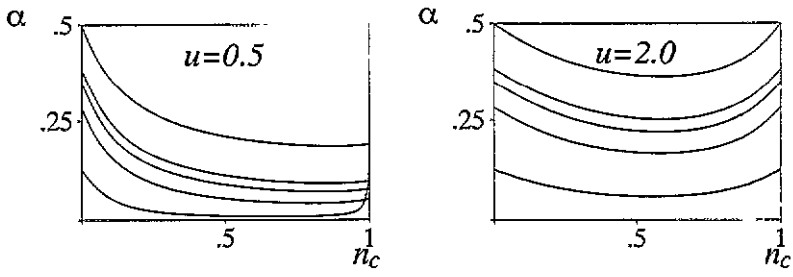

Figure 3. Exponent α characterizing the Fermi-point singularity of the momentum distribution function as function of the electron density for $N = 2, 4, 6, 8, \infty$ (bottom to top) at zero field for two values of u.

where $\theta = 2z_c^2$ varies between 2 and $2N$ as u decreases from ∞ to 0. Hence,

$$0 = \alpha(u \to 0) < \alpha < \alpha(u \to \infty) = 1/2 - 1/N + 1/2N^2 \tag{4.12}$$

showing the Luttinger liquid character of the degenerate Hubbard model at non-zero interaction. Numerical data for the exponent α are presented in figure 3.

For the density–density correlation function $\langle n_s(x,t)n_s(0)\rangle - \langle n\rangle^2$ we find contributions at wavenumbers $k = 2mP_F$ $(m = 1, \ldots, N)$ with exponents

$$2\Delta_c^{\pm} = 2(m/2N)^2\theta \qquad 2\Delta_\sigma^{\pm} = m(N-m)/N \tag{4.13}$$

arising from the choice $\Delta N_c = \Delta M_r = 0$ and $D_r = \delta_{r,N-m}$. In addition, there are $k = 0$ terms decaying as x^{-2} asymptotically. They are generated by the marginally relevant secondary operators in the conformal family of the unit operator (i.e. $\Delta N_c = \Delta M_r = 0$ and $D_c = D_r = 0$ but $N_c^{\pm}$ or one of the $N_r^{\pm}$ in equation (4.7) equal to 1)†.

From the discussion in the preceding section it is clear that the above statements are valid for any value of the coupling constant and for any filling with $n_c < 1$. For $n_c = 1$ two cases have to be discussed separately:

(i) For $n_c = 1$ and $u \leqslant u_c$ the system is in a metallic phase (described by $k_0 = k_0(u) \leqslant \pi$ in the Bethe ansatz equations). The critical exponents of the system are given by (4.7) with z_c being a function of the Coulomb interaction through (4.1). The number z_c decreases from $\sqrt{N}$ to 1 as the strength of the Coulomb interaction is varied from 0 to u_c. It is possible to expand the integral equations for the density $\rho_c(k)$ in the neighbourhood of the Mott transition $u \leqslant u_c$ to determine $k_0(u)$ from (2.6). For $N \to \infty$ the resulting expressions simplify, giving $u_c = \frac{1}{2}\sqrt{3}$ [13] and

$$z_c \approx 1 + \frac{4\sqrt{14}}{7\pi}\sqrt{1 - \frac{u}{u_c}} + \cdots. \tag{4.14}$$

The same square-root singularity in the Coulomb coupling u—but with a different numerical prefactor—is found near $u = u_c$ for finite N. Through (4.7) it also

† After completion of this work we became aware of related studies by Kawakami [30] who independently obtained the exponents $\alpha(u \to \infty)$ (equation (4.12)) and $\Delta_{c,\sigma}^{\pm}(u \to \infty, m = 1)$ (equation (4.13)).

appears in the critical exponents near the Mott transition, e.g. the exponent α of the momentum distribution (4.11) varies like

$$\alpha \approx \frac{1}{2} - \frac{4\sqrt{14}}{7\pi}\sqrt{1-\frac{u}{u_c}} + \cdots \tag{4.15}$$

as u approaches u_c from below (for $N \to \infty$).

(ii) For $u > u_c$ with one particle per site the system is in an insulating state, hence charge carrying excitations develop a gap. Excitations in the spin degrees of freedom, however, continue to be massless at zero temperature. The critical properties of this state can be described along the lines of the discussion above. For $n_c = 1$ and $u > u_c$ we have $k_0 = \pi$. Hence, the k-dependent quantities disappear from equation (3.2) leaving a system of $N-1$ coupled integral equations for the spin components of the dressed charge matrix. The expression for the conformal dimensions are of the form (3.8) and depend on the applied magnetic fields. The correlation functions for states corresponding to critical excitations are given by (3.9). These are exactly those found in the SU(N)-generalization of the Heisenberg spin chain (see e.g. [5]). Note that (3.6) implies that the momentum of the intermediate state is shifted by π for states with odd ΔM_1.

5. Magnetic field effects in the strong coupling regime

Non-zero magnetic fields h_s in (2.10) lead to finite values of the parameters Λ_s through (2.11). This effect in turn leads to a general dependence of the elements of the dressed charge matrix on the system parameters u, n_c and all of the fields. Only at and beyond certain critical values of the fields where one or more bands are completely depleted the integral equations simplify to some extent so that analytical results may become available (for a discussion of the $N = 2$ case see [8]). A more detailed study of the magnetic field dependence of the dressed charge matrix becomes possible in the strong coupling limit [9]. As in the SU(2) Hubbard model the Bethe ansatz integral equations (2.4), (2.9) and (3.2) describing the system simplify in the limit $u \to \infty$ (see also [16]). In this limit the k-dependent quantities can be eliminated which allows to study the effect of magnetic fields on the critical exponents in more detail. Upon rescaling of the variables $\lambda/u \to \tilde{\lambda} \equiv \lambda$ one obtains to leading order

$$\varepsilon_r(\lambda) = e_r^{(0)}(\lambda) - \frac{1}{2\pi}\int_{-\Lambda_r}^{\Lambda_r} \mathrm{d}\mu\, \tilde{K}_2(\lambda-\mu)\varepsilon_r(\mu) + \frac{1}{2\pi}\int_{-\Lambda_{r-1}}^{\Lambda_{r-1}} \mathrm{d}\mu\, \tilde{K}_1(\lambda-\mu)\varepsilon_{r-1}(\mu)$$
$$+ \frac{1}{2\pi}\int_{-\Lambda_{r+1}}^{\Lambda_{r+1}} \mathrm{d}\mu\, \tilde{K}_1(\lambda-\mu)\varepsilon_{r+1}(\mu) \tag{5.1}$$

($r = 1, \ldots, N-1$) where $\Lambda_0 = \Lambda_N \boxminus 0$ and

$$e_r^{(0)} = \varepsilon_r^{(0)} - [\sin 2k_0 - 2k_0]/[\pi u(1+\lambda^2)]\delta_{r,1} \equiv \varepsilon_r^{(0)} - (h_c)/(1+\lambda^2)\delta_{r,1}. \tag{5.2}$$

The integration kernels $\tilde{K}_i$ are obtained from equation (2.5) by setting $u = 1$. The dressed energy of the charged excitations is

$$\varepsilon_c(k) = -2(\cos k - \cos k_0). \tag{5.3}$$

Similarly, one finds reduced integral equations for the spin components of the density

$$\rho_r(\lambda) = \frac{n_c}{\pi(1+\lambda^2)}\delta_{r,1} - \frac{1}{2\pi}\int_{-\Lambda_r}^{\Lambda_r} \mathrm{d}\mu\, \tilde{K}_2(\lambda-\mu)\rho_r(\mu) + \frac{1}{2\pi}\int_{-\Lambda_{r-1}}^{\Lambda_{r-1}} \mathrm{d}\mu\, \tilde{K}_1(\lambda-\mu)\rho_{r-1}(\mu) + \frac{1}{2\pi}\int_{-\Lambda_{r+1}}^{\Lambda_{r+1}} \mathrm{d}\mu\, \tilde{K}_1(\lambda-\mu)\rho_{r+1}(\mu) \tag{5.4}$$

and of the dressed charge matrix ($r, s = 1, \ldots, N-1$):

$$\xi_{sr}(\lambda) = \delta_{sr} - \frac{1}{2\pi}\int_{-\Lambda_s}^{\Lambda_s} \mathrm{d}\mu\, \tilde{K}_2(\lambda-\mu)\xi_{sr}(\mu) + \frac{1}{2\pi}\int_{-\Lambda_{s-1}}^{\Lambda_{s-1}} \mathrm{d}\mu\, \tilde{K}_1(\lambda-\mu)\xi_{s,r-1}(\mu) + \frac{1}{2\pi}\int_{-\Lambda_{s+1}}^{\Lambda_{s+1}} \mathrm{d}\mu\, \tilde{K}_1(\lambda-\mu)\xi_{s,r+1}(\mu). \tag{5.5}$$

The other elements of the dressed charge matrix are found to be

$$Z_{cc} = 1 \qquad Z_{cr} = 0 \qquad r = 1, \ldots, N-1$$
$$Z_{rc} = \frac{1}{2\pi}\int_{-\Lambda_1}^{\Lambda_1} \mathrm{d}\lambda\, \tilde{K}_1(\lambda)\xi_{r1}(\lambda). \tag{5.6}$$

The expression for Z_{rc} can be rewritten using the symmetry of the kernel in (5.5) to obtain a simple relation to the densities of electrons with SU(N) index s:

$$Z_{rc} = \frac{1}{n_c}\int_{-\Lambda_r}^{\Lambda_r} \mathrm{d}\lambda\, \rho_r(\lambda) = 1 - \sum_{s=1}^{r}\frac{n_s}{n_c} \tag{5.7}$$

(since $n_s = n_c/N$ for $h_s \equiv 0$ this reproduces the corresponding entrics in equation (4.5) in the $u \to \infty$ limit).

For small magnetic fields (corresponding to large but finite values of the Λ_r) one can employ the Wiener–Hopf method to the integral equations (5.1) for the dressed energies together with condition (2.11) to compute the field dependence of the Λ_r ($g_s^-(\omega)$ is given in equation (A.9)):

$$\sin\frac{\pi r}{N}\exp\left(\frac{-\pi\Lambda_r}{N}\right) = \sum_{s,t=1}^{N-1}\left(\frac{\sin\pi rs/N\sin\pi st/N}{\sin\pi s/2N\, g_s^-(-\mathrm{i}\pi/N)}\right)\frac{\varepsilon_t^{(0)}}{\pi h_c}. \tag{5.8}$$

An analogous computation yields the actual field dependence of the Z_{rc} in equation (5.7). For $h_s \ll h_c$ we find

$$Z_{rc} = \frac{N-r}{N} - \frac{N}{\pi^2}\sum_{s=1}^{N-1} s_<(N-s_>)\frac{h_s}{h_c} \tag{5.9}$$

where $s_<$ ($s_>$) is the smaller (greater) of the integers r, s.

From (2.10) the bare energies $\varepsilon_r^{(0)}$ are known to be proportional to the applied magnetic fields, hence the Λ_r in (5.8) show the logarithmic dependence on the fields found previously in the isotropic Heisenberg spin chain [31] and the SU(2) Hubbard model [9]:

$$\Lambda_r \sim \ln(h_c/h) \tag{5.10}$$

where h is the typical strength of the fields h_r. As is known from the SU(2) Hubbard model this strong dependence on small applied fields shows up in the field dependence of the anomalous dimensions since they contain terms proportional to $1/\Lambda_s$ as the leading corrections in their spin components $\Delta_r^{\pm}$ ($r = 1, \ldots, N-1$) (see equation (4.7)) and is related to a non-analytic field dependence of the magnetic susceptibility in this system [16] and the SU(N) Heisenberg model [32]. This is in contrast to the charge components $\Delta_c^{\pm}$ of the conformal dimensions where (5.9) implies a linear dependence on the applied fields.

For sufficiently large fields ($h \simeq O(h_c)$) the system saturates in a state with all electrons occupying states in the band(s) with the lowest magnetic energy. This final state depends on the particular choice of the magnetic fields h_s in (1.1). In the following we shall consider two possible cases explicitly; generalization to others is straightforward.

One natural interpretation of the SU(N)-index is that of an orbital quantum number, i.e. taking the electrons in the rth band as having spin $S + 1 - r$ where $N = 2S + 1$. In this picture the coupling to the magnetic fields should be through Zeeman terms giving

$$h_r = -(S + 1 - r)\, h \qquad 1 \leqslant r \leqslant N = 2S + 1 \tag{5.11}$$

for the fields in (1.1) or

$$e_r^{(0)} = h - (h_c)/(1 + \lambda^2)\delta_{r,1} \tag{5.12}$$

for the bare energies (5.2). In this interpretation it is straightforward to see that the quantity h_c introduced in equation (5.2) is simply the large-u limit of the critical magnetic field beyond which the ground state of the system is ferromagnetically ordered (i.e. all electrons are in the band with spin-S). For $h > h_c$ only excitations with $\Delta M_r = 0$ ($r = 1 \ldots, N-1$) are gapless. By construction, the corresponding correlation functions are those of free spinless electrons.

Another possible interpretation of the N bands in the degenerate Hubbard model is that of degenerate bands of spin-$\pm\frac{1}{2}$ electrons. To be specific let us choose the electrons in the first N_+ bands as having spin $\uparrow$ and the ones in the remaining $N_- = N - N_+$ bands as carrying spin $\downarrow$. This choice gives

$$h_r = -h/2 \qquad \text{for } 0 \leqslant r \leqslant N_+ \qquad h_r = h/2 \qquad \text{for } N_+ < r \leqslant N \tag{5.13}$$

for the coupling of a physical magnetic field to the system corresponding to

$$e_r^{(0)} = h\delta_{r,N_+} - (h_c)/(1 + \lambda^2)\delta_{r,1}. \tag{5.14}$$

for the bare energies (5.2). Again the system saturates at large fields: for

$$h \geqslant h_s = h_c/2N_+(\psi(\tfrac{1}{2} + (1)/N_+) - \psi(\tfrac{1}{2})) \tag{5.15}$$

the ground state of the system is determined by filled bands for the ↑-electrons with densities $n_s = n_c/N_+$ while the N_- bands of ↓-electrons are empty. This state shows SU(N_+) spin symmetry, excitations involving creation of electrons in one of the ↓-bands are massive. The gapless excitations and corresponding critical exponents are given by the $u \to \infty$-limits of the expressions in the preceding section: the dressed charge matrix for the critical degrees of freedom is of the form (4.5) with N replaced by N_+ and $z_c = 1$.

Appendix

In this appendix we briefly list some mathematical results which are helpful in the solution of the Wiener–Hopf equations in sections 4 and 5. The Wiener–Hopf method itself has been reviewed in the appendix of [9].

In the reduction of the matrix Wiener–Hopf problem to scalar ones one has to diagonalize a tridiagonal $(N-1) \times (N-1)$ Toeplitz matrix of the type

$$T = \begin{pmatrix} 2y & -x & 0 & \cdots & 0 \\ -x & 2y & -x & \ddots & 0 \\ 0 & -x & 2y & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & -x & 2y \end{pmatrix} \tag{A.1}$$

where x and y are real numbers. In terms of $\lambda_\pm = y \pm \sqrt{y^2 - x^2}$ one finds for the determinant of T

$$\det T = \lambda_+^N - \lambda_-^N/\lambda_+ - \lambda_- \tag{A.2}$$

where we assumed $|y| \neq |x|$. Using this result the eigenvalues t_j of T are easily found to be

$$t_j = 2(y - x\cos(j\pi/N)) \tag{A.3}$$

with corresponding eigenvectors

$$v^{(j)} = \begin{pmatrix} v_1^{(j)} \\ \vdots \\ v_{N-1}^{(j)} \end{pmatrix} \qquad v_l^{(j)} = \sqrt{\frac{2}{N}} \sin\left(\frac{jl\pi}{N}\right). \tag{A.4}$$

Thus T is diagonalized by the matrix $U \in \mathrm{O}(N-1)$ with elements

$$U_{jl} = \sqrt{2/N}\sin(jl\pi/N). \tag{A.5}$$

The inverse of T can be written as follows

$$T^{-1} = 1/(\det T)\ S \tag{A.6}$$

where the elements of the matrix S are given by

$$S_{jl} = a_{j_<} a_{N-j_>} x^{j_>-j_<} \qquad a_j = \lambda_+^j - \lambda_-^j/\lambda_+ - \lambda_-. \tag{A.7}$$

Here again $j_< = \min(j,l)$ and $j_> = \max(j,l)$.

The main step of the Wiener–Hopf procedure is the factorization of the Fourier-transformed kernel into a product of two functions $g^{\pm}$ which are analytic in the upper and lower complex ω-plane, respectively. In the derivation of (5.8) we have used

$$2e^{-|\omega|}(\cosh\omega - \cos(s\pi/N)) = g_s^+(\omega)g_s^-(\omega) \tag{A.8}$$

with

$$g_s^+(\omega) = g_s^-(-\omega)$$
$$= 2\pi\left(-\frac{i\omega}{\pi}\right)^{-i\omega/\pi} \frac{\exp(i(\omega/\pi)(1+\ln 2))}{\Gamma((s/2N)-(i\omega/2\pi))\Gamma(1-(s/2N)-(i\omega/2\pi))} \tag{A.9}$$

where Γ denotes the gamma function. Note that the asymptotic behaviour of the functions defined in (A.9) is simply $\lim_{\omega\to\infty} g_s^{\pm}(\omega) = 1$.

Finally we want note to that in the derivation of (4.6) we do not need the explicit form of the corresponding kernel $K(\omega)$ into factors $G^{\pm}(\omega)$. One only needs the value $G^{\pm}(\omega = 0)$ which is—due to the symmetry property $G^+(\omega) = G^-(-\omega)$—equal to $\sqrt{G^+(0)G^-(0)} = \sqrt{K(0)}$ (see also [31]).

References

[1] Haldane F D M 1980 *Phys. Rev. Lett.* **45** 1358; 1981 *J. Phys. C: Solid State Phys.* **14** 2589
[2] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
[3] Cardy J L 1986 *Nucl. Phys.* B **270** 186
[4] Blöte H W J, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
Affleck I 1986 *Phys. Rev. Lett.* **56** 746
[5] Izergin A G, Korepin V E and Reshetikhin N Yu 1989 *J. Phys. A: Math. Gen.* **22** 2615
[6] Frahm H and Yu N-C 1990 *J. Phys. A: Math. Gen.* **23** 2115
[7] Woynarovich F 1989 *J. Phys. A: Math. Gen.* **22** 4243
[8] Frahm H and Korepin V E 1990 *Phys. Rev.* B **42** 10553
[9] Frahm H and Korepin V E 1991 *Phys. Rev.* B **43** 5653
[10] Kawakami N and Yang S-K 1991 *J. Phys.: Condens. Matter* **3** 5983
[11] Ogata M, Luchini M U, Sorella S and Assaad F F 1991 **66** 2388
[12] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[13] Haldane F D M 1980 *Phys. Lett.* **80A** 281; 1981 *Phys. Lett.* **81A** 545 erratum
[14] Choy T C 1980 *Phys. Lett.* **80A** 49
[15] Choy T C and Haldane F D M 1982 *Phys. Lett.* **90A** 83
[16] Lee K and Schlottmann P 1989 *Phys. Rev. Lett.* **63** 2299
[17] Lee K and Schlottmann P 1990 *Physica* B **163** 398

[18] Sutherland B 1985 *Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory* ed B S Shastry, S S Jha and V Singh (Berlin: Springer) pp 1–95
[19] Schlottmann P 1992 *Phys. Rev.* B **45** 5784
[20] Schlottmann P 1991 *Phys. Rev.* B **43** 3101
[21] Schadschneider A 1991 *Dissertation* Cologne University
[22] Read N and Newns D M 1983 *J. Phys. C: Solid State Phys.* **16** 3273
Ruckenstein A and Schmitt-Rink S 1988 *Phys. Rev.* B **38** 7188
Grilli M and Kotliar G 1990 *Phys. Rev. Lett.* **64** 1170
Czycholl C and Qin Q 1992 *Phys. Rev. Lett.* **68** 3224
[23] Lieb E H and Wu F U 1968 *Phys. Rev. Lett.* **20** 1445
[24] Klümper A, Schadschneider A and Zittartz J 1990 *Z. Phys.* B **78** 99
[25] Schadschneider A and Zittartz J 1991 *Z. Phys.* B **82** 387
[26] Sutherland B 1975 *Phys. Rev.* B **12** 3795
[27] Yang C N and Yang C P 1966 *Phys. Rev.* **150** 327
[28] de Vega H J 1988 *J. Phys. A: Math. Gen.* **21** L1089
[29] Suzuki J 1988 *J. Phys. A: Math. Gen.* **21** L1175
[30] Kawakami N 1993 *Phys. Rev.* B **47** 2928
[31] Bogoliubov N M, Izergin A G and Korepin V E 1986 *Nucl. Phys.* B **275** 687
[32] Schlottmann P 1992 *Phys. Rev.* B **45** 5293